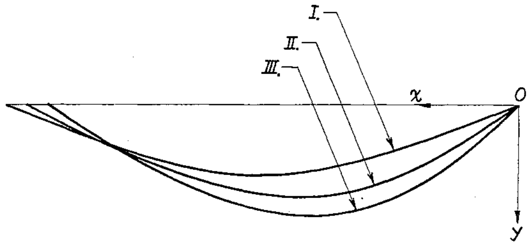

CASE	P_c/Q_0	H_0/Q_0	m	θ_1	θ_2
I.	0.244	0.796	0	22°	0°
II.	0.244	0.069	0.727	32°	4°
III.	0.226	-0.228	1.035	40°	6°

Fig. 2 Symmetric and nonsymmetric large deformation of rod with fixed length.

where b is the distance between the two ends of the rod, and its magnitude is

$$b = \int_0^l \cos\tilde{\theta}\, ds \cong lJ_0(\theta_1)J_0(\theta_2) \tag{6}$$

Performing the indicated integration in Eq. (5) with the use of Bessel series, the expression of V_0 may be put in the form

$$V_0 = q_0 l - (q_0 l^2/b)G_1 - (p_0 l^2/b)G_2 \tag{7}$$

where

$$G_1 = J_0(\theta_1)J_0(\theta_2) + \sum_{k=1}^{\infty} (-1)^k J_{4k}(\theta_1)J_{2k}(\theta_2) - \frac{4}{\pi^2}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} (-1)^{j+k} \times J_{2j-1}(\theta_1)J_{2k-1}(\theta_2) \times \left[\frac{1}{(4k+2j-3)^2} + \frac{1}{(4k-2j-1)^2}\right]$$

$$G_2 = \frac{4}{\pi^2}\left\{\sum_{k=1}^{\infty} (-1)^k \left[\frac{\pi^2}{4} J_{4k-2}(\theta_1)J_{2k-1}(\theta_2) - \frac{J_0(\theta_2)J_{2k-1}(\theta_1)}{(2k-1)^2}\right] - \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} (-1)^{j+k} J_{2j-1}(\theta_1)J_{2k}(\theta_2) \times \left[\frac{1}{(4k+2j-1)^2} + \frac{1}{(4k-2j+1)^2}\right]\right\}$$

The infinite series in Eqs. (4) and (7) converges rapidly, and only a few terms will suffice to give a satisfactory accuracy. Eliminating V_0 from Eqs. (4) by the substitution of (7) and discarding the terms having negligible contribution will yield a set of simplified simultaneous equations which can be solved either graphically or by iteration procedures.[4] The solution can be made much easier by finding the approximate value of θ_1 in advance from the following single-term condition:

$$\frac{\theta_1 P_c}{Q_0} + \left(\frac{2H_0}{Q_0} + m\right) J_1(\theta_1) - \frac{8}{\pi^2}\left\{\frac{J_0(\theta_1)}{2} + \sum_{k=1}^{\infty} (-1)^k J_{2k}(\theta_1) \times \frac{4k^2+1}{(4k^2-1)^2}\right\} = 0 \tag{8}$$

Equation (8) is obtained by carrying out the Galerkin's solution with only the first term of Eq. (2). The deformed shapes found by solving Eqs. (4) with the aid of (8) for a rod of fixed length are shown in Fig. 2.

Concluding Remarks

The two-term solution presented previously has some practical importance. One of the difficult problems encountered in industry is to determine the relationship between the internal stresses and the boundary forces of a heavy suspended elastic rod whose two supports are not on the same level. With the methods given in this note, the problem can now be treated analytically by choosing a set of inclined coordinates. Further generalization can readily be made to account for the effects of variable rigidity and to include the cases where the distributed loads are functions of s.

References

[1] Christensen, H. D., "Analysis of simply supported elastic beam columns with large deflections," J. Aerospace Sci. **29**, 1112–1121 (1962).

[2] Kantorovich, L. V. and Krylov, V. I., *Approximate Methods of Higher Analysis* (Interscience Publishers, Inc., New York, 1958), Chap. IV, p. 258.

[3] Watson, G. N., *A Treatise on the Theory of Bessel Functions* (Cambridge University Press, Cambridge, England, 1958), 2nd ed., Chap. II, p. 22.

[4] Hildebrand, F. B., *Introduction to Numerical Analysis* (McGraw-Hill Book Co., Inc., New York, 1956), Chap. 10, p. 450.

An Approximate Solution to Hypersonic Blunt-Body Problem

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Nomenclature

H = entropy gradient parameter
M = Mach number
p = $\bar{p}/\bar{\rho}_\infty \bar{U}_\infty^2$, nondimensional pressure
P_x = $\frac{1}{\sin^2\theta}\int_0^\theta p_b(\xi)\sin\xi\cos\xi\, d\xi$
R = $\bar{R}/\bar{R}_b$, dimensionless radial distance
u = $\bar{u}/\bar{U}_\infty$, dimensionless tangential velocity component
$\bar{U}_\infty$ = freestream velocity
v = $\bar{v}/\bar{U}_\infty$, dimensionless radial velocity component
α = shock inclination from direction normal to freestream
γ = ratio of specific heats
Δ = shock-layer thickness
η, κ = pressure distribution parameters
θ = angle between R and axis of symmetry
ξ = dummy integration variable
ρ = $\bar{\rho}/\bar{\rho}_\infty$, dimensionless density
ω = flow deflection angle behind shock wave

Subscripts

b = quantity evaluated at body surface
s = quantity evaluated immediately behind shock wave
1 = quantity evaluated on stagnation streamline
∞ = freestream quantity

Superscripts

* = quantity evaluated on ray through sonic point on body
— = physical quantity

I. Introduction

The approximate method outlined herein for solution of the direct hypersonic blunt body problem does not use a step-by-step advance of the solution and thus avoids the problems of error accumulation and of singular point instabilities inherent in such methods as that formulated by Belotserkovskii.[1] Integral equations are formulated ex-

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pressing conservation of mass and momentum for a volume encompassing flow in the shock layer between the stagnation streamline and rays normal to the body through the sonic line on the body. Variations of flow variables on the boundaries of this volume are described by approximating functions. Parameters in these functions are varied by iteration until an over-all consistent solution is found. The method has been tested for simple shapes such as circular cylinders and spheres with results agreeing approximately with experiment and with the results of other computational methods.

II. Analysis

The method will be described in its simplest form in which a single iterative parameter enters the function describing pressure distribution on the body, and the solution is then determined by satisfying continuity and X momentum. Referring to Fig. 1, assuming steady adiabatic inviscid flow in equilibrium and nondimensionalizing lengths by the body radius, densities, and velocities by freestream values and pressure by $\bar{\rho}_\infty \bar{U}_\infty{}^2$, leads to the following expression of conservation of mass and X momentum for the cross-hatched region:

$$(1 + \Delta)^2 \sin\theta = 2 \int_1^{1+\Delta} \rho u R dR \quad (1)$$

$$-(1 + p_\infty) \frac{(1 + \Delta)^2}{2} + \frac{1}{\sin^2\theta} \int_0^\theta p_b(\xi) \cos\xi \sin\xi d\xi + \int_1^{1+\Delta} (p + \rho u^2) R dR - \cot\theta \int_1^{1+\Delta} \rho u v R dR = 0 \quad (2)$$

By expressing integrands as integrable functions, the equations may be reduced to algebraic form. For the integrals across the shock layer, the integrands are approximated by polynomials whose coefficients are defined in terms of function values and their derivatives on the boundaries. For second- and third-degree polynomial approximations, the integration formulas are, respectively,

$$\int_1^{1+\Delta} f(R) dR = \Delta \left[\frac{2}{3} f_b + \frac{1}{3} f_s + \frac{\Delta}{6}\left(\frac{\partial f}{\partial R}\right)_b\right] \quad (3)$$

$$\int_1^{1+\Delta} f(R)\, dR = \frac{\Delta}{2}\left\{f_b + f_s + \frac{\Delta}{6}\left[\left(\frac{\partial f}{\partial R}\right)_b - \left(\frac{\partial f}{R \partial}\right)_s\right]\right\} \quad (4)$$

Representing pressure distribution on the body by

$$p_b(\theta)/p_{b1} = \cos^2\kappa\theta + \eta \sin^2\kappa\theta \quad (5)$$

allows evaluation of the integral

$$\frac{1}{\sin^2\theta} \int_0^\theta p_b(\xi) \sin\xi \cos\xi d\xi = P_x = \frac{p_{b1}}{4}(1 + \eta) + \frac{p_{b1}}{8}(1 - \eta)\left[\frac{1 - \cos(2 + 2\kappa)\theta}{(2 + 2\kappa)\sin^2\theta} + \frac{1 - \cos(2 - 2\kappa)\theta}{(2 - 2\kappa)\sin^2\theta}\right] \quad (6)$$

To reduce this to a one-parameter pressure distribution, η is selected as

$$\eta = (p_\infty/p_{b1})/[\sin^2(\kappa\pi/2)] \quad (7)$$

Approximating shock inclination by a second-degree polynomial in θ,

$$\alpha = \left(\frac{d\alpha}{d\theta}\right)_1 \theta + \left[\frac{\alpha^*}{\theta^{*2}} - \frac{1}{\theta^*}\left(\frac{d\alpha}{d\theta}\right)_1\right]\theta^2 \quad (8)$$

allows evaluation of the geometrical relationship

$$ln \frac{(1 + \Delta^*)}{(1 + \Delta_1)} = \int_0^{\theta^*} \tan(\theta - \alpha) d\theta \quad (9)$$

to give a relation between shock-layer thickness at the sonic point and the stagnation streamline detachment distance.

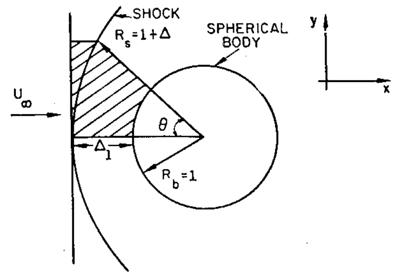

Fig. 1 Schematic diagram of coordinate system and flow geometry.

The basic conservation equations are transformed to algebraic form using these relations and two sets of equations derived, the first valid on the stagnation streamline, and the second valid on rays passing through the sonic line of the body. Iteration is used between these two sets of equations until an over-all consistent solution is attained. Use of a third-degree polynomial approximation for the stagnation region and a second-degree for the sonic region seems to optimize simplicity and accuracy.

To evaluate derivatives, use is made of the basic differential equations of motion. In the axisymmetric case, the nonvanishing entropy gradient at the body must be considered. Hayes and Probstein[2] discuss this feature and show that, for a sphere

$$(\partial u/\partial R)_b = H p_b \sin\theta - u_b \quad (10)$$

where

$$H = \{[(\rho_s - 1)^2/p_s\rho_s(1 + \Delta)](d\alpha d\theta_1)\}_1 \quad (11)$$

Using the second-order integration formula and incorporating sonic point properties leads to the following form for the continuity and X momentum equations:

$$\Delta^2[2\rho_s u_s + \rho_b u_b + \rho_b{}^2 u_b{}^2 H \sin\theta - 3\sin\theta] + \Delta[4\rho_b u_b + 2\rho_s u_s - 6\sin\theta] - 3\sin\theta = 0 \quad (12)$$

$$4\Delta(p_b + \rho_b u_b{}^2) + 2\Delta(1 + \Delta)(p_s + \rho_b u_s{}^2) + \Delta^2(p_b + \rho_b u_b{}^2)(1 + \rho_b u_b H \sin\theta) - 2\Delta(1 + \Delta)\rho_s u_s v_s \cot\theta + \Delta^2 \rho_b u_b{}^2 \cot^2\theta + 6P_x - 3(1 + p_\infty)(1 + \Delta)^2 = 0 \quad (13)$$

All of the flow properties at the sonic point on the body are well defined. All of the flow properties at the shock on the sonic ray are fully determined by the shock inclination α^*. For a given θ^*, there are two independent unknown quantities α^* and Δ^*, and Eqs. (12) and (13) suffice to determine these quantities. A solution for α^* cannot be expressed explicitly; hence, iteration must be used.

For the stagnation streamline solution using third-degree polynomial approximations, an equation of state must be specified. An ideal gas assumption leads to the following relations expressing conservation of mass and X momentum as $\theta \to 0$,

$$\left(\frac{d\omega}{d\theta}\right)^2 \left\{\frac{2\gamma\rho_s p_s}{\gamma\rho_s p_s - 1} + \frac{\rho_b p_b}{\rho_s p_s}\frac{(\rho_s - 1)^2}{(1 + \Delta)^2} \Big/ \left(\frac{d\omega}{d\alpha}\right)^2\right\} = + \frac{(\Delta^2 + 6\Delta + 6)}{\Delta^2}\frac{p_b - p_s}{v_s} + \frac{4p_b(1 - \eta)\kappa^2}{v_s} - \frac{6}{\Delta}[2\rho_b p_b(1 - \eta)\kappa^2]^{1/2} \quad (14)$$

$$\frac{6}{\Delta}[2\rho_b p_b(1 - \eta)\kappa^2]^{1/2} = \frac{6(1 + \Delta)^2}{\Delta^2} - \frac{6 + 5\Delta}{\Delta} \times \left(1 + \frac{d\omega}{d\theta}\right) - \rho_b H p_b + \left(1 + \frac{d\omega}{d\theta}\right)\left(2\frac{d\omega}{d\theta}\right) \Big/ (1 - \gamma\rho_s p_s) + \frac{d\omega}{d\theta}\left(1 + \frac{d\omega}{d\theta}\right) - \frac{4\rho_s}{(\gamma + 1)}\left(\frac{d\omega}{d\theta}\right)^2 \Big/ \left(\frac{d\omega}{d\alpha}\right)^2 + \left(1 - \frac{d\omega}{d\theta}\Big/\frac{d\omega}{d\alpha}\right)\left(2\frac{d\omega}{d\theta}\right)\Big/(1 - 1/\gamma\rho_s p_s) \quad (15)$$

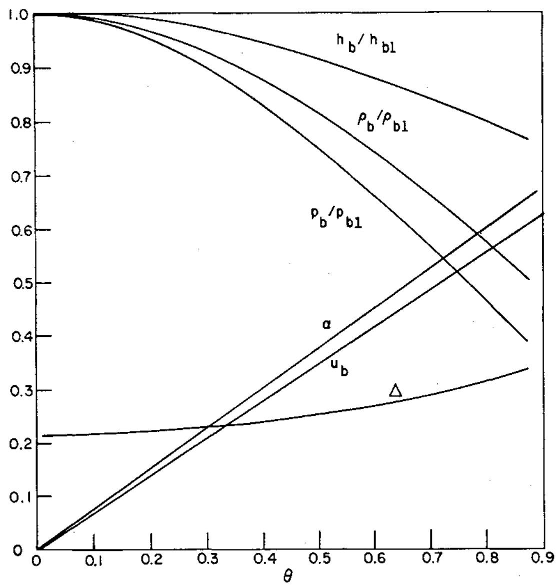

Fig. 2 Flow variable profiles for an ideal gas, $M_\infty = 3.0$, $\gamma = 1.4$.

Values of flow variables on the body and behind the shock on the stagnation streamline are found readily from normal shock conservation relations. With a trial value of Δ_1, Eqs. (14) and (15) can be solved for the unknowns κ and $(d\omega/d\theta)_1$. These quantities directly determine θ^* and $(d\alpha/d\theta)_1$.

The iteration cycle may be described as follows. A trial value of Δ_1 is selected and substituted into Eqs. (14) and (15), giving θ^* and $(d\alpha/d\theta)_1$. Using this θ^*, Eqs. (12) and (13) are solved for α^* and Δ^*. Equation (9) is then used to calculate a corresponding Δ_1. The difference between the assumed value and the value resulting from the calculation cycle becomes the error associated with the trial Δ_1. By iteration, the two values are made to agree to the required degree of accuracy.

III. Applications and Results

Computer programs were prepared using this method and computations accomplished on the UNIVAC LARC computer. The rapidity of these calculations may be emphasized by the fact that solutions for an ideal gas were obtained for twenty Mach numbers from 1.7 to 100, and these solutions listed in both tabular and graphical form in less than 1 min of computer execution time. Figure 2 is a typical result for a freestream Mach number of 3. Figure 3 compares the pressure distribution with that of Belotserkovskii[1] and the experimental results of Baer.[3] Table 1 gives a comparison of detachment distances.

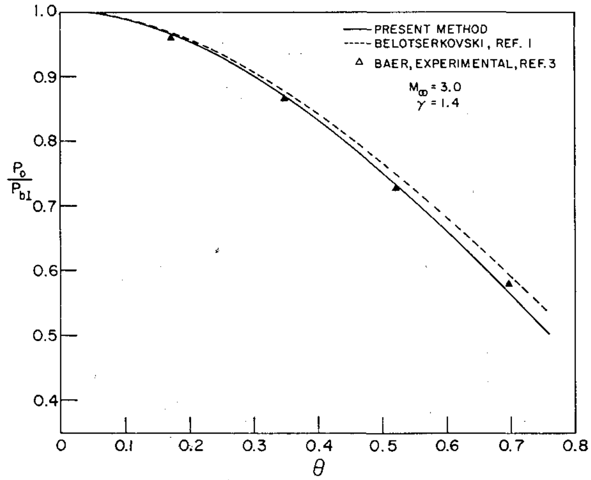

Fig. 3. Pressure distribution on a sphere for an ideal gas.

Table 1 Comparison of shock detachment distance for a spherical body ($\gamma = 1.4$)

M_∞	Present method	Belotserkovskii
1.7	0.472	...
1.8	0.418	...
2.0	0.350	...
2.5	0.260	...
3.0	0.216	0.215
4.0	0.175	0.175
6.0	0.147	0.149
8.0	0.138	...
10.0	0.134	0.136
15.0	0.130	...
20.0	0.128	...
30.0	0.127	...
50.0	0.127	...
100.0	0.127	...
∞	0.127	0.128

This method is readily adaptable to arbitrary equations of state. The solutions strongly depend upon property values behind a normal shock and their stagnation values and upon property values at the sonic point on the body. For an equation of state of the general form

$$p = p(\rho, h) \tag{16}$$

these quantities may be determined readily and the oblique shock conservation equations solved by iteration. For example, the convenient Bade[4] approximation for the equation of state of dissociated air was fitted into these programs and computations readily accomplished.

References

[1] Belotserkovskii, O. M., "The calculation of flow over axisymmetric bodies with a decaying shock wave," Avco Corp. Rept. RAD-TM-62-64 (September 1962).

[2] Hayes, W. D. and Probstein, R. F., *Hypersonic Flow Theory* (Academic Press, Inc., New York, 1959), Chap. VI, p. 202.

[3] Baer, A. L., "Pressure distributions on a hemisphere cylinder at supersonic and hypersonic Mach numbers," Arnold Engineering Development Center, AEDC TN-61-96 (1961).

[4] Bade, W. L., "Simple analytical approximation to the equation of state of dissociated air," ARS J. **29**, 298–299 (1959).

Flow Fields about Highly Yawed Cones by the Inverse Method

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Nomenclature

B_s = bluntness parameter
C_p = pressure coefficient
h = altitude
M_∞ = freestream Mach number
R_s = conic radius of curvature at X axis (Fig. 1)
V_∞ = freestream velocity
α = angle of attack
δ = cone half-angle
θ, φ = spherical coordinates

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